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Homotopy perturbation for conservative Helmholtz–Duffing oscillators

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Abstract

The approximate periodic solutions of the Helmholtz–Duffing oscillator are obtained by homotopy perturbation. The Helmholtz–Duffing oscillator becomes a Duffing oscillator when the homotopy parameter degenerates to one and a Helmholtz oscillator when it is zero. Since the behaviors of the solutions in the positive and negative directions are quite different, the asymmetric equation is separated into two auxiliary equations. The auxiliary equations are solved by homotopy perturbation method. A new analytical period for the Helmholtz–Duffing equation is derived. The resulting second-order approximate periodic solutions are compared to the analytical solutions using numerical integration with improved accuracy over some existing methods. Thus, the homotopy perturbation is very effective for the asymmetric nonlinear oscillators.

1. Introduction

In science and engineering, there are many phenomena and applications related to asymmetric oscillators. Mickens [1] and Hu [2,3] examined the periodic solutions of quadratic nonlinear oscillators (QNO) and mixed parity nonlinear oscillator using perturbation method and harmonic balance method respectively. Some one dimensional structural systems with an initial curvature [4] can be simulated by a Helmholtz–Duffing oscillator [5–8]. These systems include shallow arches, ship roll dynamics, some electrical circuits, microperforated panel absorber and heavy symmetric gyroscope [9–12]. We shall solve the following conservative Helmholtz–Duffing oscillator by the homotopy perturbation method:

$$\ddot{u} + u + (1-\sigma)u^2 + \sigma u^3 = 0$$

with initial conditions

$$u(0) = a \quad \text{and} \quad \dot{u}(0) = 0 \tag{1}$$

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where σ is an asymmetric parameter representing the extend of asymmetry and an over dot denotes differentiation with respect to t. When $\sigma = 1$, Eq. (1) is a classical Duffing oscillator. Eq. (1) becomes a Helmholtz oscillator with a single-well potential when $\sigma = 0$. The QNO studied by Hu [2,3] is essentially a Helmholtz oscillator with a single-well potential. For the Helmholtz–Duffing oscillator (1), we obtain the second-order approximate periods and the corresponding periodic solutions using homotopy perturbation method proposed by He [13–15]. Strictly speaking, homotopy perturbation is not a perturbation method which requires a small parameter in the differential equation but is an artificial perturbation method constructed by embedding an artificial parameter $p \in [0, 1]$ in the equation so that a complicated equation can be solved from the already known solution of another equation by unfolding the parameter in small perturbation steps. This technique yields a very rapidly converging series of the solution, thus it can provide an effective and convenient solution method for nonlinear differential equations.

The organization of the paper is as follows. In Section 2, we introduce two auxiliary equations which are necessary as asymmetric solutions having different behaviors in positive and negative directions. In Section 3, we construct two new homotopy equations based on the auxiliary equations and obtain second-order approximations for the periods and the corresponding periodic solutions. In Section 4, we give some numerical comparisons of the obtained second-order approximations of the periods with the newly introduced analytical periods. Finally, we provide some conclusions in Section 5.

In the paper, we give an explicit formula for the analytical period of the Helmholtz–Duffing oscillator. When degenerated to the QNO for the initial amplitude $a = 0.49$ as a comparison example, the analytical period is 9.2080. Hu's result [2] is 8.8118(−4 percent error) which is much better than Micken's 6.9816(−24 percent). Ours 9.2840(0.825 percent) is even better in all studied cases. Our method can be applied to a much wider class of problems too.

2. Auxiliary equations of Helmholtz–Duffing oscillator

Introducing a new time variable $\tau = \omega t$, the Helmholtz–Duffing oscillator becomes

$$\omega^2 u'' + u + (1 - \sigma)u^2 + \sigma u^3 = 0 \tag{2}$$

$$u(0) = a, \quad u'(0) = 0 \tag{3}$$

where ω is an unknown angular frequency to be determined and a prime denote differentiation with respect to τ. That the behaviors of an asymmetric nonlinear oscillator are different in positive and negative directions suggests that the equation can be conveniently studied in two parts [14,16]. Eq. (2) is equivalent to the following two auxiliary equations:

$$\omega^2 u'' + u + (1 - \sigma)u^2 \operatorname{sgn}(u) + \sigma u^3 = 0 \quad \text{for } u \geqslant 0 \tag{4}$$

$$\omega^2 u'' + u - (1 - \sigma)u^2 \operatorname{sgn}(u) + \sigma u^3 = 0 \quad \text{for } u \leqslant 0 \tag{5}$$

in which

$$\operatorname{sgn}(u) = \begin{cases} 1, & u > 0 \\ 0, & u = 0 \\ -1, & u < 0 \end{cases}$$

is a sign function.

Assume that the system oscillates between asymmetric limits zone $[-b, a]$, for positive a and b. When $u = a$ and $u = -b$, one has $u' = 0$, where a is given by the initial condition (1) and b is an unknown amplitude in the negative direction to be determined here. Multiply u' on both sides of Eq. (2):

$$\omega^2 u'' u' + uu' + (1 - \sigma)u^2 u' + \sigma u^3 u' = 0 \tag{6}$$

Integrating once, we have the first integral

$$\tfrac{1}{2}\omega^2 u'^2 + \tfrac{1}{2}u^2 + \tfrac{1}{3}(1 - \sigma)u^3 + \tfrac{1}{4}\sigma u^4 = C \tag{7}$$

where C is an integration constant. The Helmholtz–Duffing oscillator (2) is a conservative system. Substituting the conditions that $u' = 0$ when $u = a$ and $u = -b$, one has

$$\tfrac{1}{2}a^2 + \tfrac{1}{3}(1-\sigma)a^3 + \tfrac{1}{4}\sigma a^4 = \tfrac{1}{2}b^2 - \tfrac{1}{3}(1-\sigma)b^3 + \tfrac{1}{4}\sigma b^4 \tag{8}$$

Solving for b, we have an exact solution:

$$b = \frac{1}{9\sigma}(3a\sigma + 4 - 4\sigma) + \frac{1}{9\sigma}\Delta^{1/3} - \frac{2}{9\sigma}(9a^2\sigma^2 + 6a\sigma - 6a\sigma^2 + 43\sigma - 8 - 8\sigma^2)\Delta^{-1/3} \tag{9}$$

where

$$\begin{aligned}\Delta = {} & 270a^2\sigma^2(1 + a\sigma - \sigma) - 72a\sigma(1+\sigma^2) - 516\sigma(1-\sigma) + 64(1-\sigma^3) + 630a\sigma^2 + 54\sigma[-12(1+\sigma^2) \\ & + 16a(a + 1 + a\sigma^4 - \sigma^3) + 78\sigma(1 + a^3\sigma - a^3\sigma^2) - 8a^3\sigma(1-\sigma^3) - 172a^2\sigma(1+\sigma^2) - 120a\sigma(1-\sigma) \\ & + 9a^4\sigma^2(1 + 5a\sigma + 10\sigma + 3a^2\sigma^2 - 5a\sigma^2 + \sigma^2) + 447a^2\sigma^2]^{1/2}\end{aligned}$$

Therefore, the two auxiliary equations (4) and (5) with initial conditions (3) become

$$\omega^2 u'' + u + (1-\sigma)u^2\,\mathrm{sgn}(u) + \sigma u^3 = 0 \quad \text{for } u \geqslant 0 \tag{10}$$

$$u(0) = a \quad \text{and} \quad u'(0) = 0$$

$$\omega^2 u'' + u - (1-\sigma)u^2\,\mathrm{sgn}(u) + \sigma u^3 = 0 \quad \text{for } u \leqslant 0 \tag{11}$$

$$u(0) = b \quad \text{and} \quad u'(0) = 0$$

We shall introduce the homotopy perturbation method to solve these two auxiliary equations, one at a time below.

3. Homotopy perturbation solutions of the auxiliary equations

For Eq. (10), let ω_{a0} be the initial angular frequency. We construct a homotopy:

$$(1-p)\omega_{a0}^2(u'' + u) + p[\omega^2 u'' + u + (1-\sigma)u^2\,\mathrm{sgn}(u) + \sigma u^3] = 0 \tag{12}$$

where $p \in [0,1]$ is an embedding parameter, $u = u(\tau,p)$, and $\omega = \omega(p)$. When $p = 0$, Eq. (12) is simple harmonic:

$$u'' + u = 0, \quad u(0) = a \quad \text{and} \quad u'(0) = 0 \tag{13}$$

For $p = 1$, we have the auxiliary equation (10). $u(\tau,1)$ is therefore the required solution of Eq. (10). As the embedding parameter p increases from 0 to 1, the solutions $u(\tau,p)$ and $\omega(p)$ of the homotopy equation (12) change from the initial approximation $u_{a0}(\tau)$ and ω_{a0} to the required solutions $u(\tau)$ and ω of Eq. (10).

The basic assumption of the technique is that the solution of Eq. (12) can be obtained as a power series of the homotopy perturbation parameter p:

$$u(\tau) = u_0(\tau) + u_1(\tau)p + u_2(\tau)p^2 + \cdots \tag{14}$$

$$\omega = \omega_{a0} + \omega_{a1}p + \omega_{a2}p^2 + \cdots \tag{15}$$

Substituting Eqs. (14) and (15) into Eq. (12) and equating the terms with identical powers of the embedding parameter p, we can obtain a series of linear equations. The initial zeroth approximation is given by

$$u_0'' + u_0 = 0 \tag{16}$$

with initial conditions $u_0(0) = a$ and $u_0'(0) = 0$; the first approximation is given by

$$\omega_{a0}^2(u_1'' + u_1 - u_0'' - u_0) + \omega_{a0}^2 u_0'' + u_0 + (1-\sigma)u_0^2\,\mathrm{sgn}(u) + \sigma u_0^3 = 0 \tag{17}$$

with initial conditions $u_1(0) = 0$ and $u_1'(0) = 0$; the second approximation is given by

$$\omega_{a0}^2(u_2'' + u_2 - u_1'' - u_1) + \omega_{a0}^2 u_1'' + u_1 + 2\omega_{a0}\omega_{a1}u_0'' + 2(1-\sigma)u_0u_1\,\mathrm{sgn}(u) + 3\sigma u_0^2 u_1 = 0 \tag{18}$$

with initial conditions $u_2(0) = 0$ and $u_2'(0) = 0$, and so on.

The solution of Eq. (16) is simply

$$u_{a0}(\tau) = a\cos\tau \tag{19}$$

Substituting Eq. (19) into Eq. (17) gives

$$\omega_{a0}^2(u_1'' + u_1) = -[-\omega_{a0}^2 a\cos\tau + a\cos\tau + (1-\sigma)a^2\cos^2\tau\,\mathrm{sgn}(a\cos\tau) + \sigma a^3\cos^3\tau] \tag{20}$$

It should be pointed that the sign function can be expanded in trigonometry series $\mathrm{sgn}(a\cos\tau) = \mathrm{sgn}(\cos\tau) = (4/\pi)\sum_{i=0}^{+\infty}(-1)^i/(2i+1)\cos[(2i+1)\tau]$. When taking three terms,

$$\mathrm{sgn}(\cos\tau) \approx \frac{4}{\pi}\left(\cos\tau - \frac{1}{3}\cos 3\tau + \frac{1}{5}\cos 5\tau\right) \tag{21}$$

Since there should not be secular terms in u_1, contributions from $\cos\tau$ on the right hand side of Eq. (20) must be eliminated; therefore,

$$\omega_{a0} = \sqrt{1 + \frac{3}{4}\sigma a^2 + \frac{8a}{3\pi}(1-\sigma)} \tag{22}$$

The corresponding approximate period of the oscillator is

$$T_{a1} = \frac{2\pi}{\omega_{a0}} = 2\pi\left[1 + \frac{3}{4}\sigma a^2 + \frac{8a}{3\pi}(1-\sigma)\right]^{-1/2} \tag{23}$$

And, the solution to Eq. (20) is

$$u_{a1}(\tau) = (-A - B - C)\cos\tau + A\cos^3\tau + B\cos^5\tau + C\cos^7\tau \tag{24}$$

where

$$A = \frac{a^2}{8\pi\omega_{a0}^2}\left(\pi a\sigma - \frac{32}{9}\sigma + \frac{32}{9}\right),\quad B = \frac{19a^2}{45\pi\omega_{a0}^2}(\sigma - 1),\quad C = \frac{4a^2}{15\pi\omega_{a0}^2}(1-\sigma)$$

Substituting Eqs. (19), (22) and (24) into Eq. (18) gives

$$\omega_{a0}^2(u_2'' + u_2) = -[\omega_{a0}^2(-u_1'' - u_1) + \omega_{a0}^2 u_1'' + u_1 + 2\omega_{a0}\omega_{a1}u_0'' + 2(1-\sigma)u_0u_1\,\mathrm{sgn}(u) + 3\sigma u_0^2 u_1] \tag{25}$$

The sign function $\mathrm{sgn}(u)$ is given in Eq. (21).

Similarly, to eliminate secular terms in u_2 of Eq. (25), one needs to annihilate the coefficient of $\cos\tau$ giving

$$\begin{aligned}\omega_{a1} = \frac{1}{2\omega_{a0}a}&\left[(1-\omega_{a0}^2)\left(-A - B - C + \frac{3}{4}A + \frac{5}{8}B + \frac{35}{64}C\right)\right.\\ &\left.+\frac{2(1-\sigma)}{\pi}\left(-\frac{491}{480}aC - \frac{199}{240}aB - \frac{8}{15}aA\right) + 3\sigma a^2\left(-\frac{1}{8}A - \frac{13}{64}B - \frac{33}{128}C\right)\right]\end{aligned} \tag{26}$$

The second approximate period of the oscillator is

$$T_{a2} = \frac{2\pi}{\omega_{a0} + \omega_{a1}} \tag{27}$$

Likewise, one considers the auxiliary equation (11) and constructs a homotopy for the initial angular frequency ω_{b0}:

$$(1-p)\omega_{b0}^2(u'' + u) + p[\omega^2 u'' + u - (1-\sigma)u^2\,\mathrm{sgn}(u) + \sigma u^3] = 0 \tag{28}$$

Without repeating the above process, we just give the results of the first two approximate periods

$$u_{b0}(\tau) = b\cos\tau \tag{29}$$

$$\omega_{b0} = \sqrt{1 + \frac{3}{4}\sigma b^2 - \frac{8b}{3\pi}(1-\sigma)} \tag{30}$$

$$T_{b1} = 2\pi\left[1 + \frac{3}{4}\sigma b^2 - \frac{8b}{3\pi}(1-\sigma)\right]^{-1/2} \tag{31}$$

$$u_{b1}(\tau) = (-\hat{A} - \hat{B} - \hat{C})\cos\tau + \hat{A}\cos^3\tau + \hat{B}\cos^5\tau + \hat{C}\cos^7\tau \tag{32}$$

$$\begin{aligned}\omega_{b1} = \frac{1}{2\omega_{b0}b}\Bigg[&(1-\omega_{b0}^2)\left(-\hat{A} - \hat{B} - \hat{C} + \frac{3}{4}\hat{A} + \frac{5}{8}\hat{B} + \frac{35}{64}\hat{C}\right) - \frac{2(1-\sigma)}{\pi}\\ &\times\left(-\frac{491}{480}b\hat{C} - \frac{199}{240}b\hat{B} - \frac{8}{15}b\hat{A}\right) + 3\sigma b^2\left(-\frac{1}{8}\hat{A} - \frac{13}{64}\hat{B} - \frac{33}{128}\hat{C}\right)\Bigg]\end{aligned} \tag{33}$$

where

$$\hat{A} = \frac{b^2}{8\pi\omega_{b0}^2}\left(\pi b\sigma + \frac{32}{9}\sigma - \frac{32}{9}\right), \quad \hat{B} = \frac{19b^2}{45\pi\omega_{b0}^2}(1-\sigma), \quad \hat{C} = \frac{4b^2}{15\pi\omega_{b0}^2}(\sigma - 1)$$

and

$$T_{b2} = \frac{2\pi}{\omega_{b0} + \omega_{b1}} \tag{34}$$

4. Results and discussion

The first approximate period T_1 and the corresponding periodic solution $u_0(t)$ of Eqs. (2) and (3) are, respectively,

$$T_1 = \frac{T_{a1} + T_{b1}}{2} \tag{35}$$

$$u_0(t) = \begin{cases} a\cos\omega_{a0}t, & 0 \leqslant t \leqslant \dfrac{T_{a1}}{4} \\ b\cos\omega_{b0}\left(t - \dfrac{T_{a1}}{4} + \dfrac{T_{b1}}{4}\right), & \dfrac{T_{a1}}{4} \leqslant t \leqslant \dfrac{T_{a1}}{4} + \dfrac{T_{b1}}{2} \\ a\cos\omega_{a0}\left(t + \dfrac{T_{a1}}{2} - \dfrac{T_{b1}}{2}\right), & \dfrac{T_{a1}}{4} + \dfrac{T_{b1}}{2} \leqslant t \leqslant T_1 \end{cases} \tag{36}$$

The second approximate period T_2 and the corresponding periodic solution $u_0(t) + u_1(t)$ of Eqs. (2) and (3) are, respectively,

$$T_2 = \frac{T_{a2} + T_{b2}}{2} \tag{37}$$

$$u_0(t) + u_1(t) = \begin{cases} a\cos[(\omega_{a0} + \omega_{a1})t] + u_{a1}[(\omega_{a0} + \omega_{a1})t], & 0 \leqslant t \leqslant \dfrac{T_{a2}}{4} \\ b\cos\left[(\omega_{b0} + \omega_{b1})\left(t - \dfrac{T_{a2}}{4} + \dfrac{T_{b2}}{4}\right)\right] + u_{b1}\left[(\omega_{b0} + \omega_{b1})\left(t - \dfrac{T_{a2}}{4} + \dfrac{T_{b2}}{4}\right)\right], & \dfrac{T_{a2}}{4} \leqslant t \leqslant \dfrac{T_{a2}}{4} + \dfrac{T_{b2}}{2} \\ a\cos\left[(\omega_{a0} + \omega_{a1})\left(t + \dfrac{T_{a2}}{2} - \dfrac{T_{b2}}{2}\right)\right] + u_{a1}\left[(\omega_{a0} + \omega_{a1})\left(t + \dfrac{T_{a2}}{2} - \dfrac{T_{b2}}{2}\right)\right], & \dfrac{T_{a2}}{4} + \dfrac{T_{b2}}{2} \leqslant t \leqslant T_2 \end{cases} \tag{38}$$

The exact period T_e of the Helmholtz–Duffing oscillator is

$$T_e = \int_0^a \frac{2dx}{\sqrt{a^2 - x^2 + \frac{2}{3}(1-\sigma)(a^3 - x^3) + \frac{1}{2}\sigma(a^4 - x^4)}} + \int_0^b \frac{2dx}{\sqrt{b^2 - x^2 - \frac{2}{3}(1-\sigma)(b^3 - x^3) + \frac{1}{2}\sigma(b^4 - x^4)}} \tag{39}$$

where b is given in Eq. (9).

For comparison, the exact period T_e is obtained by integrating Eq. (39) and the approximate periods T_1 and T_2 are given in Eqs. (35) and (37), respectively. The relative errors are defined as $((T - T_e)/T_e) \times 100$. When $\sigma = 0$, the Helmholtz–Duffing equation degenerates to a QNO. Hu [2] gave the solutions of QNO by the harmonic balance method and the result of T_1 in Eq. (35) is in exact agreement with his. The values of a should satisfy $a < 0.5$, this is because that if $a = 0.5$, the corresponding QNO has a homoclinic orbit with period $+\infty$. Table 1 compares the approximate periods with the corresponding exact period for $\sigma = 0$. It indicates that T_2

Table 1
Comparison of the approximate periods with the exact period for $\sigma = 0$.

a	T_e	T_1 (percent error)	T_2 (percent error)
0.10	6.311599	6.311242 (−0.005656)	6.311687 (0.0014)
0.20	6.411392	6.409514 (−0.02929)	6.411861 (0.007315)
0.30	6.629357	6.622552 (−0.1026)	6.631084 (0.02605)
0.40	7.124567	7.096187 (−0.3983)	7.131872 (0.1025)
0.45	7.706476	7.627741 (−1.022)	7.726369 (0.2581)
0.46	7.905170	7.801416 (−1.312)	7.930979 (0.32648)
0.47	8.167157	8.023325 (−1.7611)	8.201926 (0.4257)
0.48	8.545167	8.327834 (−2.543)	8.594657 (0.5791)
0.49	9.207997	8.811815 (−4.303)	9.283958 (0.8249)

Table 2
Comparison of approximate periods with the exact period for $\sigma = 0.5$.

a	T_e	T_1 (percent error)	T_2 (percent error)
0.01	6.283133	6.283132 (−0.00000159)	6.283133 (0.00000031018)
0.02	6.282974	6.282971 (−0.000004775)	6.282975 (0.000001592)
0.05	6.281851	6.281831 (−0.0003184)	6.281856 (0.00007959)
0.10	6.277721	6.277642 (−0.0012584)	6.2777408 (0.000315)
0.20	6.259970	6.259659 (−0.004968)	6.260044 (0.00118)
0.40	6.174673	6.173590 (−0.017539)	6.174871 (0.0032066)
0.60	5.999096	5.996744 (−0.0392059)	5.999168 (0.00120018)
0.80	5.728173	5.722892 (−0.092193)	5.727689 (−0.008449)
1.0	5.395022	5.3842845 (−0.199026)	5.393792 (−0.022798)
2.0	3.841120	3.8025801 (−1.00335)	3.8385182 (−0.067735)
5.0	1.894337	1.8588624 (−1.87267)	1.8926895 (−0.086968)
10	1.004172	0.98321194 (−2.087298)	1.0032703 (−0.089795)
20	0.5143428	0.50328477 (−2.14993)	0.51387692 (−0.090578)
50	0.2082766	0.20375994 (−2.168587)	0.20808755 (−0.0907687)
100	0.1045213	0.1022518 (−2.171328)	0.1044264 (−0.090795)

Table 3
Comparison of approximate periods with the exact period for $\sigma = 0.9$.

a	T_e	T_1 (percent error)	T_2 (percent error)
0.01	6.2829757	6.2829757 (−0.00000005)	6.2829757 (0.000000013)
0.10	6.2622091	6.26220029 (−0.0001407)	6.2622098 (0.000011178)
0.50	5.8065177	5.80388099 (−0.04541)	5.8065145 (−0.00005511)
1	4.841329	4.82413578 (−0.3551)	4.8411819 (−0.003038)
5	1.5054557	1.47638384 (−1.9310)	1.5043714 (−0.072025)
10	0.7727359	0.7564511 (−2.1074)	0.77207482 (−0.08555)
50	0.1561833	0.1527948 (−2.1696)	0.15604183 (−0.090579)
100	0.0781394	0.07644255 (−2.1716)	0.07806849 (−0.090748)

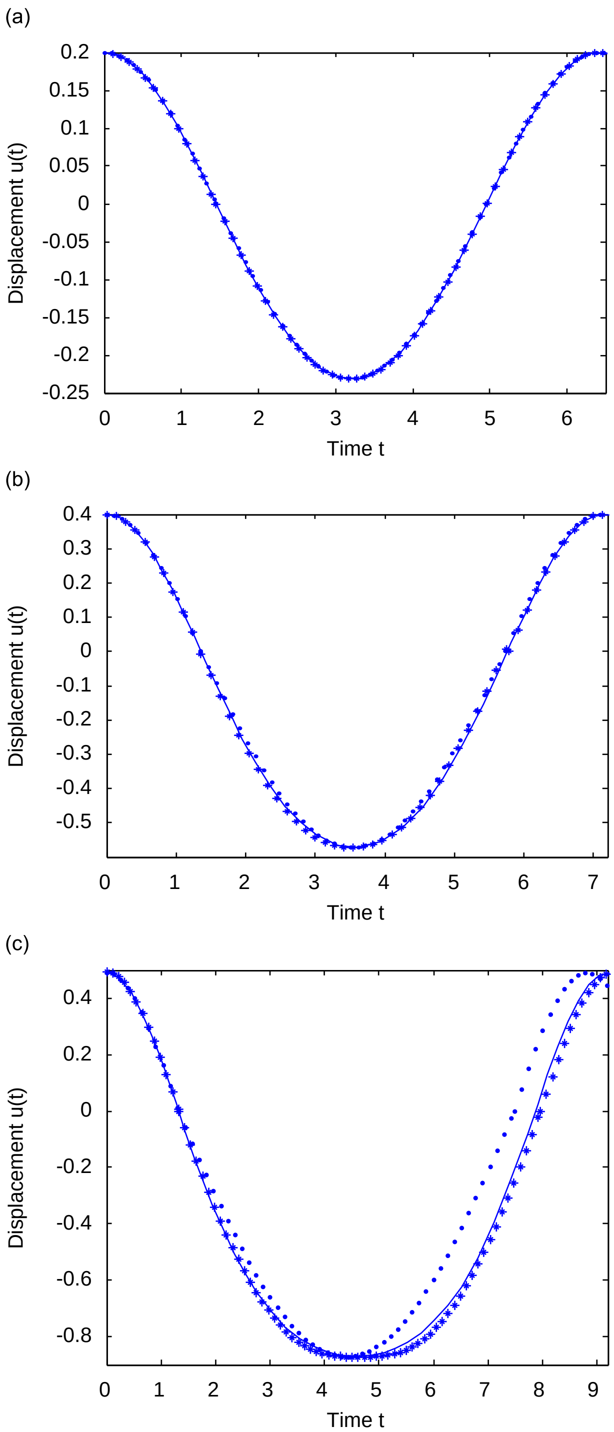

Fig. 1. Comparison of the approximate solutions with numerical solution for $\sigma = 0$. Numerical: –; $u_0(t)$: …; $u_0(t) + u_1(t)$:*; (a) $a = 0.2$; (b) $a = 0.4$; (c) $a = 0.49$.

is more accurate than T_1 in general. As the initial values a increases, the relative error of the approximate periods T_1 and T_2 increase. When $a = 0.49$, the relative error of T_1 with respect to T_e is less than 4.304 percent, whereas, the relative error of T_2 with respect to T_e is less than 0.825 percent. For general Helmholtz–Duffing

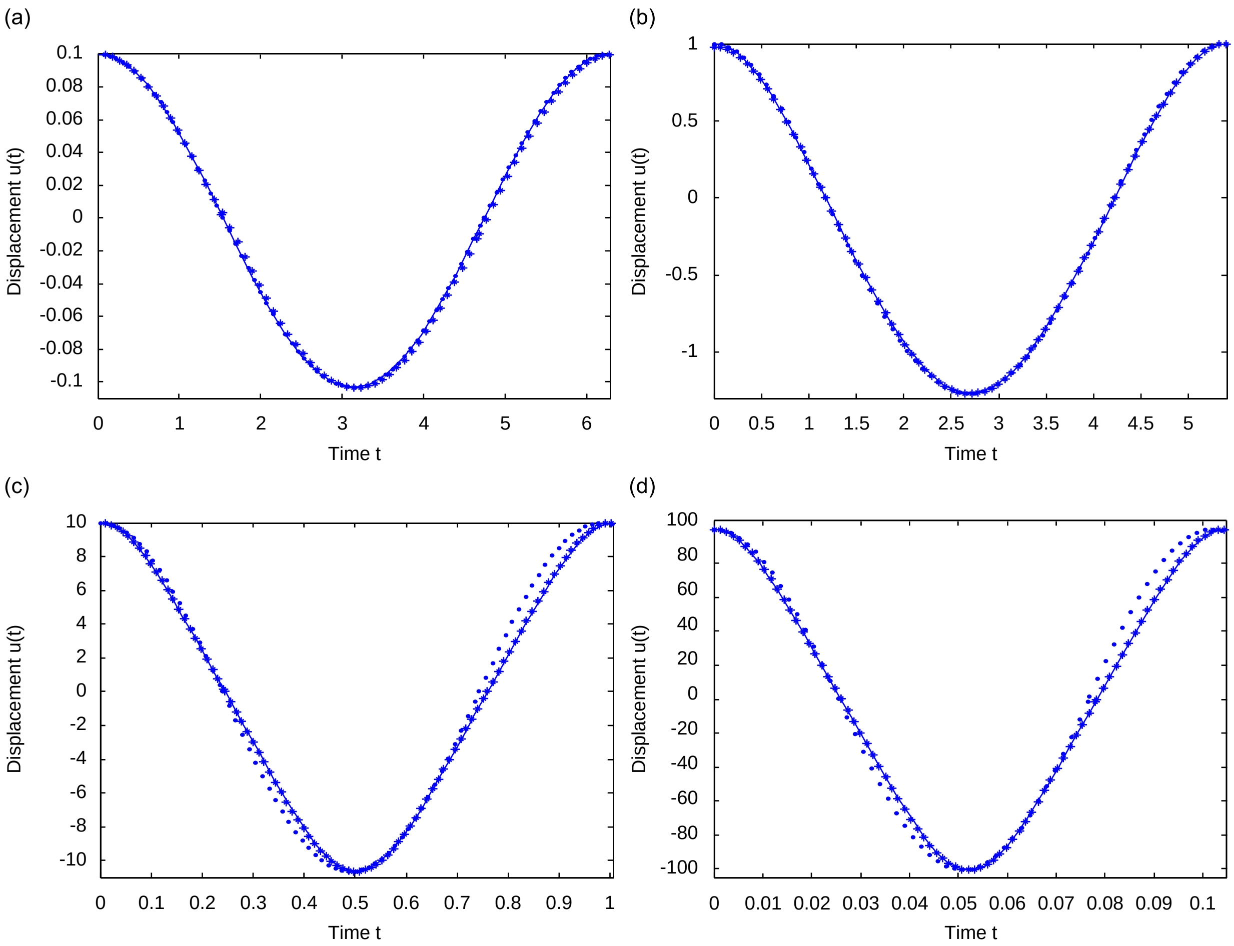

Fig. 2. Comparison of the approximate solutions with numerical solution for $\sigma = 0.5$. Numerical: –; $u_0(t)$: ...; $u_0(t) + u_1(t)$:*; (a) $a = 0.1$; (b) $a = 1$; (c) $a = 10$; (d) $a = 100$.

oscillator, $\sigma \neq 0$. Tables 2 and 3 compare the approximate periods with the corresponding exact periods for $\sigma = 0.5$ and 0.9. From Tables 2 and 3, it is evident that T_2 is more accurate than T_1. As the initial value a increases, the relative error of approximate periods T_1 and T_2 also increase. Even when $a = 100$, the relative error of T_2 with respect to T_e is less than 0.091 percent, that is, the approximate period T_2 is very accurate.

Figs. 1–3 give some comparisons of the approximate solutions with numerical solutions. Figs. 1(a)–(c) compare the approximate solutions $u_0(t)$ and $u_0(t) + u_1(t)$ with the numerically integrated solution for $\sigma = 0$ and initial conditions $a = 0.2$, 0.4, 0.49 while Fig. 2 are for $\sigma = 0.5$ and $a = 0.1$, 1, 10 and 100 and Fig. 3 are for $\sigma = 0.9$ and $a = 1$, 10, 100, respectively. It is shown that $u_0(t) + u_1(t)$ is very accurate in all cases.

When the Helmholtz–Duffing oscillator degenerates to the Duffing oscillator by letting $\sigma = 1$, the results are also compared favorably with those of Belendez et al. [17].

5. Conclusions

We have applied the homotopy perturbation method to obtain approximate expression for the periods and corresponding to periodic solutions of the Helmholtz–Duffing oscillator. First, the asymmetric oscillator is separated into two auxiliary equations whose restoring forces are odd functions. The amplitude in the negative direction is obtained by conversation of energy. Then, the auxiliary equations are approximately solved by constructing homotopy perturbation. The first-order approximate period T_1 and periodic solution $u_0(t)$ are in

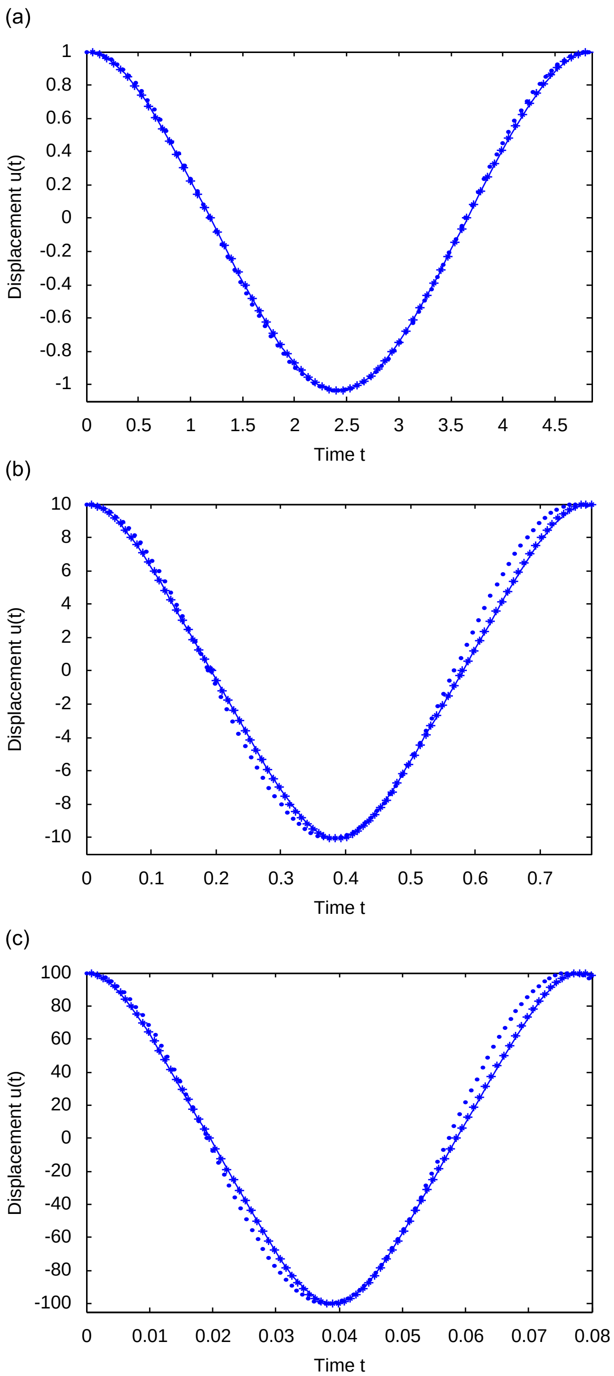

Fig. 3. Comparison of the approximate solutions with numerical solution for $\sigma = 0.9$. Numerical: –; $u_0(t)$: . . .; $u_0(t) + u_1(t)$:*; (a) $a = 1$; (b) $a = 10$; (c) $a = 100$.

agreement with those obtained by using harmonic balance method. The second-order approximate period T_2 and the periodic solution $u_0(t) + u_1(t)$ are more accurate than the first-order approximation for each asymmetric parameter σ. For very large initial amplitude $a = 100$, the relative error of T_2 with respect to the analytical period T_e is less than 0.091 percent. Thus, the homotopy perturbation method is very effective and convenient for some asymmetric nonlinear differential equations.

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